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ADDENDUM

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Abstract
We present other special functions $s(t)$ leading to the extensions of the principal symmetry algebra (infinite-dimensional) which were not included in our article (Güngör F 2001 *J. Phys. A: Math. Gen.* **34** 4313–4321).

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Recently, we investigated symmetry properties for the two-dimensional generalized Burgers equation [1]

$$(u_t + uu_x - u_{xx})_x + s(t)u_{yy} = 0 \qquad s(t) \neq 0. \tag{1}$$

We showed that, for an arbitrary $s(t)$, equation (1) is invariant under an infinite-dimensional Lie algebra whose general element is represented by

$$V = X(f) + Y(g) \tag{2a}$$

$$X(f) = f(t)\partial_x + f'(t)\partial_u \tag{2b}$$

$$Y(g) = g(t)\partial_y - \frac{g'(t)}{2s(t)}y\partial_x - \left(\frac{g'(t)}{2s(t)}\right)' y\partial_u \tag{2c}$$

where $f(t)$ and $g(t)$ are arbitrary smooth functions and the primes denote time derivatives. The commutation relations are

$$\begin{gathered}[X(f_1), X(f_2)] = 0 \qquad [X(f), Y(g)] = 0 \\ [Y(g_1), Y(g_2)] = X\left(\frac{1}{2s}(g_1'g_2 - g_1g_2')\right)\end{gathered} \tag{3}$$

where [. , .] is the Lie bracket. In particular, we showed that the above algebra is further extended in the cases:

Case 1. $s(t) = \sigma t^{\alpha}$, σ = constant (case I in [1]).

Case 2. $s(t) = \sigma e^{\alpha t}$, σ = constant (case II in [1]).

In this addendum, we would like to point out that, in addition to the above cases, there are further specific functions $s(t)$ for which the symmetry algebra contains one additional basis element. The functions $s(t)$ and corresponding additional generators are presented below:

Case 3. $s(t) = \sigma(1+t^2)^{-3/2}\exp(2\alpha\arctan t)$:

$$C_1 = xt\partial_x + \alpha y\partial_y + (1+t^2)\partial_t + (x-tu)\partial_u. \tag{4}$$

The nonzero commutation relations satisfy

$$[X(f), C_1] = X(tf - (1+t^2)f') \qquad [Y(g), C_1] = Y(\alpha g - (1+t^2)g'). \tag{5}$$

Case 4. $s(t) = \sigma t^{-3}\exp(\alpha/t)$, $\alpha \neq 0$:

$$C_2 = xt\partial_x - \tfrac{1}{2}\alpha y\partial_y + t^2\partial_t + (x-tu)\partial_u. \tag{6}$$

The nonzero commutation relations satisfy

$$[X(f), C_2] = X(tf - t^2 f') \qquad [Y(g), C_2] = -Y(\tfrac{1}{2}\alpha g + t^2 g'). \tag{7}$$

The subcase $\alpha = 0$ is exceptional in that there are two additional generators:

$$C = xt\partial_x + t^2\partial_t + (x-tu)\partial_u \qquad D = D_{-3} = x\partial_x - \tfrac{3}{2}y\partial_y + 2t\partial_t - u\partial_u \tag{8}$$

with nonzero commutators

$$\begin{aligned} &[C, D] = -2C \qquad [X(f), C] = X(tf - t^2 f') \qquad [Y(g), C] = -Y(t^2 g') \\ &[X(f), D] = X(f - 2tf') \qquad [Y(g), D] = Y(-\tfrac{3}{2}g - 2tg'). \end{aligned} \tag{9}$$

Case 5. $s(t) = \sigma(1-t^2)^{-3/2}\left(\dfrac{1+t}{1-t}\right)^{\alpha}$:

$$C_3 = xt\partial_x - \alpha y\partial_y + (t^2-1)\partial_t + (x-tu)\partial_u. \tag{10}$$

The nonzero commutation relations satisfy

$$[X(f), C_3] = X(tf + (1-t^2)f') \qquad [Y(g), C_3] = Y(-\alpha g + (1-t^2)g'). \tag{11}$$

Case 6. $s(t) = \sigma t^{\alpha}(t+\beta)^{-\alpha-3}$, $\alpha \neq -3$, $\beta \neq 0$:

$$C_4 = C + \tfrac{1}{2}\beta D_\alpha \qquad D_\alpha = x\partial_x + (\alpha + \tfrac{3}{2})y\partial_y + 2t\partial_t - u\partial_u \tag{12}$$

and C is as in equation (8). The nonzero commutation relations satisfy

$$\begin{aligned} &[X(f), C_4] = X\big((t + \tfrac{1}{2}\beta)f - t(t+\beta)f'\big) \\ &[Y(g), C_4] = Y\big(\tfrac{1}{4}\beta(3+2\alpha)g - t(t+\beta)g'\big). \end{aligned} \tag{13}$$

We remark that this list completes all possible forms of $s(t)$ extending (2*a*) by one or two generators. Using the commutation relations (5), (7), (9), (11) and (13) we can easily verify that a general element of the symmetry algebra $X(f)+Y(g)+pC_a$ $(a = 1, 2, 3, 4)$ is conjugate, under the adjoint action of the symmetry group of the equation, to C_a if $p \neq 0$, and either to $Y(g)$ or $X(f)$ otherwise (see [1] for details). Moreover, requiring that C_a be embedded into a two-dimensional Lie subalgebra of the symmetry algebra, reductions of (1) with the new coefficients $s(t)$ listed above (cases 3–6) to ordinary differential equations (actually second order) can be performed and hence new exact solutions invariant under two-dimensional Lie point groups can be constructed.

Reference

[1] Güngör F 2001 Symmetries and invariant solutions of the two-dimensional variable coefficient Burgers equation *J. Phys. A: Math. Gen.* **34** 4313–21